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Saylor Academy awards

Dan Rimniceanu

this certificate of achievement for

MA111: Introduction to Mathematical Reasoning

19 aprilie 2018
Issue Date

11475279
Certificate ID

MA111: Introduction to Mathematical Reasoning

Unit 1: Logic
In this unit, you will begin by considering various puzzles, including Ken-Ken and Sudoku. You will learn the importance of tenacity in approaching mathematical problems including puzzles and brain teasers. You will also learn why giving names to mathematical ideas will enable you to think more effectively about concepts that are built upon several ideas. Then, you will learn that propositions are (English) sentences whose truth value can be established. You will see examples of self-referencing sentences which are not propositions. You will learn how to combine propositions to build compound ones and then how to determine the truth value of a compound proposition in terms of its component propositions. Then, you will learn about predicates, which are functions from a collection of objects to a collection of propositions, and how to quantify predicates. Finally, you will study several methods of proof including proof by contradiction, proof by complete enumeration, etc.

Completing this unit should take you approximately 31 hours.

- Upon successful completion of this unit, you will be able to:
    - determine the truth value of a proposition;
    - use truth tables to verify the logical equivalence of two mathematical expressions, statements, and propositions, as well as to determine the category of a proposition: tautology, contingency, or contradiction;
    - form the negation, contrapositive, converse, and inverse of a proposition;
    - express statements formally using universal quantifiers, logical connectives, and predicates;
    - use the technique "proof by contradiction” to verify a mathematical statement; and
    - construct counterexamples to verify the falsity of a mathematical statement in particular contexts.

- **1.1: Sudoku and Latin Squares**
    - Wikipedia: "Sudoku"URL

      Read this article. Try not to get sidetracked looking at variations. Pay special attention to the growth of the number of Latin squares as the size increases. Note that if you want to look ahead at the type of problem you will be asked to solve, check the file "Logic.pdf" at the end of Unit 2.
    - Tom Davis' "The Mathematics of Sudoku"URL

      Read Tom Davis' paper, paying special attention to the way he names the cells and to his development of language. Next, if you have not done Sudoku puzzles before, Web Sudoku and Daily Sudoku and are two popular sites. Do one or two before moving on to Ken-Ken.

- Wikipedia: "Latin Square"URL

  Read this article on Latin Squares.

- **1.2: Ken-Ken**
  - Harold Reiter's "Introduction to Mathematical Reasoning"URL

    This article is optional. If you have an interest in solving Ken-Ken problems, then you will find this section interesting. Otherwise, omitting it will not hinder your understanding of subsequent material. Read this article for an introduction to Ken-Ken and complete the exercises in the PDF.

  - *The New York Times*: "Ken-Ken Puzzles"URL

    This activity is optional. Attempt to complete one of these puzzles. Note that you can choose the level of difficulty (easier, medium, and harder). After a few practices, challenge yourself to attempt a Ken-Ken puzzle that is at the next level of difficulty. Do not allow yourself to get addicted!

- **1.3: SET**
  - Set Enterprises: "Daily Puzzle"URL

    This activity is optional. Read the game rules by clicking on the "daily puzzle rules" link, and play a bit.

- **1.4: Other Brain Teasers**
  - Khan Academy: "Brain Teasers"URL

    Pick out a few videos to watch on brain teasers. The puzzle will be introduced to you at the beginning of the video. You should pause the video and attempt to solve the puzzle before viewing the solution. Watch the solutions only if you absolutely cannot solve the puzzle; then, go back and reattempt the problem.

- **1.4.1: Truth Tellers and Liars**
  - University of Chicago: Antonio Montalban and Yannet Interian's "Module on Puzzles"URL

    Work on the problems on this webpage: liars and truth-tellers puzzles, the Rubik's cube, knots and graphs, and arithmetic and geometry.

- **1.4.2: Coin Weighing Puzzles**
  - Alexander Bogomolny's "A Fake Among Eight Coins"URL

Problems about finding the counterfeit coin among a large group of otherwise genuine coins are quite abundant. Attempt to solve the problem on this webpage. Solutions appear at the bottom of the webpage. If this type of logical thinking interests you, attempt to find similar problems to solve with an online search.

- **1.5: Propositional Logic**
  - Hofstra University: Stefan Waner and Steven R. Costenoble's "Introduction to Logic"URL

    Read this page. This text will enable you to see the very close connection between propositional logic and naïve set theory, which you will study in Unit 3.

  - Indian Institute of Technology Madras: Dr. Kamala Krithivasan's "Propositional Logic"Page

    Watch this lecture. In particular, focus on the information provided from the 12-minute mark until the 18-minute mark. In this lecture, you will learn which sentences are propositions.

  - **1.5.1: Compound Proposition**
    - The University of Aukland: "Boolean Operators"Page

      Watch this video, which will help you later when you are asked to build proofs of statements about rational numbers and about integers.

    - Mark Thorsby's "Introduction to Propositional Logic Part I"Page

      Watch this video, which will help you later when you are asked to build proofs of statements about rational numbers and about integers.

    - Wikipedia: "Logical Connective"URL

      Read this article, which covers the properties of connectives. While reading, pay special attention to the connection between the Boolean connective and its Venn diagram.

  - **1.5.2: Truth Tables**
    - University of Cincinnati, Blue Ash: Kenneth R. Koehler's "Logic and Set Theory"URL
      - Logical Operations and Truth Tables
      - Properties of Logical Operators
      - Arguments
      - Boolean Algebra

      Read these four sections of Koehler's lectures on logic and set theory. A contingency is simply a proposition that is caught between tautology (at the top) and contradiction (at the bottom). In other words, it is a proposition which is true for some values of its components and false for others. For example "if it rains today, it will snow tomorrow" is a contingency, because it can be true or false depending on the truth values of the two component propositions.

- Indian Institute of Technology Madras: Dr. Kamala Krithivasan's "Propositional Logic, Continued"Page

  Watch this lecture.

- **1.6: Predicate Logic**

  - Indian Institute of Technology Madras: Dr. Kamala Krithivasan's "Predicates and Qualifiers"Page

    Watch these lectures.

- **1.6.1: Modus Ponens and Modus Tollens**

  - Indian Institute of Technology Madras: Dr. Kamala Krithivasan's "Methods of Proof"Page

    Watch this lecture.

- **1.6.2: Proofs by Contradiction**

  - California State University, San Bernardino: Peter Williams' "Notes on Methods of Proof"URL

    Read the following sections: "Introduction", "Definition and Theorems", "Disproving Statements", and "Types of Proofs". The types of proofs include Direct Proofs, Proof by Contradiction, Existence Proofs, and Uniqueness Proofs. You may stop the reading here; we will cover the sixth one, Mathematical Induction, later in the course.

- **1.6.3: Problem Solving Strategies**

  - Old Dominion University: Shunichi Toida's "Problem Solving"URL

    Read through the examples in the article. The problems are not difficult, but they do serve as clear illustrations of the various aspects of entry-level problem solving.

- **1.6.4: Contrapositive and Equivalent Forms**

  - Gowers' Weblog: "Basic Logical Relationships between Statements, Converses, and Contrapositives"URL

    Read this article, paying special attention to the parts of converses and contrapositives.

Unit 2: Sets

In this unit, you will explore the ideas of what is called 'naive set theory.' Contrasted with 'axiomatic set theory,' naive set theory assumes that you already have an intuitive understanding of what it means to be a set. You should mainly be concerned with how two or more given sets can be combined to build other sets and how the number of members (i.e. the cardinality) of such sets is related to the cardinality of the given sets.

Completing this unit should take you approximately 10 hours.

- Upon successful completion of this unit, you will be able to:
  - formally define a set using set builder notation;
  - identify and explain the difference between being a member of a set and being a subset of a set;
  - build new sets from existing ones using the set operations, including unions, intersections, complements, symmetric difference, Cartesian product, and relative complements;
  - verify properties of sets (e.g., distributive laws, DeMorgan's laws, etc.);
  - compute the power set of a given set;
  - compute the cardinality of the union (intersection) of two finite sets given the cardinalities of the sets and their intersection (union); and
  - determine if two sets are *equivalent*.

- **2.1: What Is a Set? Set Builder Notation**
  - Indian Institute of Technology Madras: Dr. Kamala Krithivasan's "Sets"Page

    Watch this lecture, which defines sets and will familiarize you with set notation and set language.

  - Old Dominion University: Shunichi Toida's "Set Theory"URL
    - Introduction to Set Theory
    - Representation of Sets
    - Basics of Sets
    - Mathematical Reasoning

    Read these pages, which discuss the basics of set theory. Note that there are three ways to define a set. The third method, recursion, will come up again later in the course, but this is a great time to learn it.

  - **2.1.1: The Empty Set, the Universal Set**
    - University of California, San Diego: Edward Bender and S. Williamson's "Arithmetic, Logic and Numbers, Unit SF: Sets and Functions"URL

      Read pages SF-1 through SF-8 for an introduction to sets, set notation, set properties and proofs, and ordering sets.

- Kasidej Anchaleenukoon's "Set Theory"Page

  Watch this video for an elementary introduction to set theory. This will be useful to you in case you feel uneasy about the reading above.

- **2.1.2: Sets with Sets as Members**
  - University of California, San Diego: Edward Bender and S. Williamson's "Arithmetic, Logic and Numbers, Unit SF: Sets and Functions"URL

    Read pages SF-9 through SF-11 to learn about subsets of sets. This text also is useful for learning how to prove various properties of sets.

- **2.2: Building New Sets from Given Sets**
  - Old Dominion University: Shunichi Toida's "Set Operations"URL

    Read this page, then test your understanding by working the four problems at the bottom.

- **2.2.1: Properties of Union, Intersection, and Complementation**
  - Old Dominion University: Shunichi Toida's "Properties of Set Operation"URL
    Read this page. It is important that you become aware that sets combine under union and intersection in very much the same ways that numbers combine under addition and multiplication. For example, $A \cup B=B \cup A A \cup B=B \cup A$ is a way to say union is commutative in the same way as $x+y=y+x x+y=y+x$ says addition is commutative. One difference, however, is that the properties of addition and multiplication are defined as part of the number system (in our development) whereas the properties of sets under the operations we have defined are provable and hence must be proved.
  - Simpson College: Lydia Sinapova's "Boolean Algebra"URL

    Read this lecture, paying special attention to the definition of Boolean Algebra and to the isomorphism between the two systems of propositional logic and that of sets. Work the three exercises at the bottom of the PDF and then have a look at the solutions at the end of the document.

- **2.2.2: The (Boolean) Algebra of Subsets of a Set**
  - The University of Western Australia: Greg Gamble's "Set Theory, Logic, and Boolean Algebra"URL

    Read this lecture.

- **2.2.3: Using Characteristic Functions to Prove Properties of Sets**
  - Jerusalem College of Technology: Dr. Dana-Picard's "The Characteristic Function of a Set"URL

Read this page. This brief text will show you how to use characteristic functions to prove properties of sets. However, there are other reasons to learn how to do this. You will see later in the course that functions (not just characteristic functions play a critical role in the theory of cardinality (set size).

- **2.3: The Cartesian Product of Two or More Sets**

  - Jerusalem College of Technology: Dr. Dana-Picard's "The Characteristic Function of a Set"URL

    Read this page for a definition and overview of the Cartesian product of sets.

- **2.3.1: The Disjoint Union and Addition**

  - American Public University: "Disjoint Sets"Page

    Watch this brief lecture on disjoint sets.

- **2.3.2: The Cartesian Product and Multiplication**

  - Nikos Drakos and Ross Moore's "Cartesian Product of Sets"URL

    Read this page, paying special attention to the proof of proposition 3.3.3 at the end of the page. There is a nice proof of this using characteristic functions, which you will be asked to produce later in the course.

- **2.4: Counting Finite Sets**
- **2.4.1: The Cardinality of the Power Set of a Set**

  - American Public University: "Equivalent Sets"Page

    Watch this lecture, which discusses equivalent sets.

- **2.4.2: The Formula |A| + |B| = |A ∪ B| + |A ∩ B|**

  - University of Hawaii: G.N. Hile's "Set Cardinality"URL

    Read this page, which demonstrates the basic inclusion/exclusion equation outlined in the title of this subunit. The examples on this webpage are especially interesting; pay attention to example 2, which is about playing cards.

Unit 3: Introduction to Number Theory

This unit is primarily concerned with the set of natural numbers $N=\{0,1,2,3,\ldots\}$N={0,1,2,3,...}. The axiomatic approach to NN will be postponed until the unit on recursion and mathematical induction. This unit will help you understand the multiplicative and additive structure of NN. This unit begins with integer representation: place value. This fundamental idea enables you to completely understand the algorithms we learned in elementary school for addition, subtraction, multiplication, and division of multi-digit integers. The beautiful idea in the Fusing Dots paper will enable you to develop a much deeper understanding of the representation of integers and other real numbers. Then, you will learn about the multiplicative building blocks, the prime numbers. The Fundamental Theorem of Arithmetic guarantees that every positive integer greater than 1 is a prime number or can be written as a product of prime numbers in essentially one way. The Division Algorithm enables you to associate with each ordered pair of non-zero integers - a unique pair of integers - the quotient and the remainder. Another important topic is modular arithmetic. This arithmetic comes from an understanding of how remainders combine with one another under the operations of addition and multiplication. Finally, the unit discusses the Euclidean Algorithm, which provides a method for solving certain equations over the integers. Such equations with integer solutions are sometimes called Diophantine Equations.

Completing this unit should take you approximately 32 hours.

- Upon successful completion of this unit, you will be able to:
    - express integers and rational numbers using base b notation, where b is an integer not equal to 0 or 1;
    - state, identify, and interpret the meaning of certain famous number theory conjectures, including the Twin Prime Conjecture, Goldbach's Conjecture, the Riemann Hypothesis, the Fundamental Theorem of Arithmetic, and the Fundamental Theorem of Algebra;
    - solve problems related to the digits of a number;
    - find the prime factorization of composite numbers and use such factorizations to find the GCDs and the LCMs of two or more given integers;
    - use modular arithmetic to find remainders;
    - use the Euclidean Algorithm to solve integer linear equations in two unknowns;
    - solve quadratic equations in Z_6, Z_7, and Z_{11}; and
    - solve basic Diophantine equations.

- **3.1: Place Value Notation**
    - Harold Reiter's "Fusing Dots"URL

      Read this essay, paying special attention to the exercises at the end. You may find the second half of this reading very difficult. You can access the solutions for selected problems here. Don't worry about understanding all of the details your first time through the reading. Instead, concentrate on the material in the first five sections of the document, and then attempt to generally understand the subsequent sections on Fusing Dots.

    - Wisconsin Technical College System: Laurie Jarvis' "Understanding Place Value"URL

      Review this presentation. This information will most likely serve as a review of place value.

- **3.2: Prime Numbers**

  - University of St. Andrews: J.J. O'Connor and E.F. Robertson's "Prime Numbers"URL

    Read this page, which includes a good overview of prime numbers and also a list of unsolved problems. Pay special attention to the unsolved problems 1 and 2.

  - Wikipedia: "Prime Number"URL

    Read this entry on prime numbers, which will give you an idea of the connections between number theory and other areas of mathematics.

- **3.2.1: An Infinitude of Primes**

  - The University of Tennessee at Martin: Chris K. Caldwell's "Euclid's Proof of the Infinitude of Primes"URL

    Read the entire webpage on Euclid's proof of the infinitude of primes. Be sure you understand why the prime PP is not already in the list of primes; if necessary, re-read this text a few times until you have fully grasped this concept.

- **3.2.2: Conjectures about Primes**

  - The University of Utah: Peter Alfeld's "Prime Number Problems"URL

    Read this page to get an idea of some of the many unsolved problems about prime numbers.

- **3.2.2.1: The Twin Prime Conjecture**

  - *Plus Magazine*: "Mathematical Mysteries: Twin Primes"URL

    Read this page. Take note of the definition of Brun's constant. Also note that this is related to the Intel's famous $475 million recall of Pentium chips. Please also feel free to click on the link to "Enumeration to 1e14 of the twin primes and Brun's constant" link at the end of the page to read associated content.

- **3.2.2.2: Goldbach's Conjecture**

  - Mike James' "Goldbach Conjecture: Closer to Solved?"URL

    Read this brief article to learn about the Goldbach conjecture. Problems like this are the subject of intense work by mathematicians around the world, and progress is made nearly every year towards solving them.

- **3.2.2.3: The Riemann Hypothesis**

  - Wikipedia: "Riemann Hypothesis"URL

    Read this article about the Riemann Hypothesis.

- **3.3: Fundamental Theorem of Arithmetic (FTA)**
    - Khan Academy: "The Fundamental Theorem of Arithmetic"Page

      Watch this brief video, which provides an informative, though far less technical, introduction to the Fundamental Theorem of Arithmetic.

    - Alexander Bogomolny's "Euclid's Algorithm" and "GCD and the Fundamental Theorem of Arithmetic"URL
        - Euclid's Algorithm
        - GCD and the Fundamental Theorem of Arithmetic

      Read these pages for information about the Fundamental Theorem of Arithmetic (FTA). Please note that we are going to postpone the proof of FTA until the end of Unit 4.

    - Wikipedia: "Fundamental Theorem of Arithmetic"URL

      Read this entire article on the fundamental theorem of arithmetic. The article may take more time to read than some others.

    - University of California, Berkeley: Zvezdelina Stankova-Frenkel's "Unique and Nonunique Factorization"URL

      Read this page. In particular, focus on the exercise in the reading. Do not be intimidated by the notation in the essay, just read it down to the part on ideals.

- **3.4: Modular Arithmetic, the Algebra of Remainders**
    - The Mathsters: "Modular Arithmetic"Page

      Watch these lectures, which address the concepts outlined earlier. Then, if you chose to work through the Ken-Ken material in subunit 1.2, go to section 9 of the paper "Using Ken-Ken to Build Reasoning Skills" from subunit 1.2, and re-read the section to recall how to use modular arithmetic as a strategy for Ken-Ken puzzles.

- **3.4.1: Division by 3, 9, and 11**
    - Dr. James Tanton's "Divisibility by 3, 9, and 11"Page

      Watch these videos, which will help you understand the divisibility rules for 3, 9, and 11.

- **3.4.2: Building the Field Z?**
    - Harold Reiter's "Building the Rings to Z_6 and Z_7"URL

Read this entire essay, paying special attention to understanding the operations ⊕ and ⊗ (read 'oplus' and 'otimes') in Z? and Z?. Then, work the problems 1 and 4 on Z??. You may check your solutions here.

- 3.4.3: **Square Roots in Modular Arithmetic**
- **3.4.3.1: The Addition of Remainders**
  - The Art of Problem Solving: "2000 AMC 12 Problems"URL

  Try to solve the problem before checking the solution. This problem asks: what is the units' digit of the 2012th Fibonacci number? See if you can work this using your understanding that remainders work perfectly with respect to addition. After you have attempted this problem, review the solution on this page.

- **3.4.3.2: The Multiplication of Remainders**
  - By now, the following type of problem should be familiar: what is the units digit of the expression $7^{2012} \times 13^{2011}$? See if you can work this using your understanding that remainders work perfectly with respect to multiplication. In other words, if you know the remainder when N is divided by d, then you can find the remainder when N^3 is divided by d.

    The solution to this question is mentioned below, but please only check it after you have attempted the problem.

    Solution: The solution to the initial problem mentioned above is that the remainder when N^3 is divided by d is the same as when r^3 is divided by d, where r is the remainder when N is divided by d.

- **3.5: Functions in Number Theory**

- **3.5.1: The Floor and the Ceiling Functions**
  - The University of Western Australia: Greg Gamble's "The Floor or Integer Part Function" and "Number Theory 1"URL
    - The Floor or Integer Part Function
    - Number Theory 1

  Read these lectures.

- **3.5.2: The Greatest Common Divisor (GCD) and Least Common Multiple (LCM) of Two Integers**
  - Andy Schultz's "GCD and LCM"URL

  Read this page, paying special attention to the relationship between the GCD and LCM.

- **3.5.3: The Sigma-Function, Summing Divisors**

- Wikipedia: "Divisor Function"URL

  Read this article about the sum of the divisors of a number.

- Harold Reiter's "Just the Factors, Ma'am"URL

  Read this article, paying special attention to Sections 3 and 4, where you will learn about geometry of the divisors of an integer. Complete the problems on the document above, and then check your answers here.

- **3.6: The Euclidean Algorithm**

  - Alexander Bogomolny's "The Euclidean Algorithm"URL
    - Euclid's Algorithm
    - An Interactive Illustration
    - Euclid's Game
    - Binary Euclid's Algorithm
    - GCD and the Fundamental Theorem of Arithmetic

    Read each of these pages for information on the Euclidean Algorithm.

  - Michael Slone, Kimberly Lloyd, and Chi Woo's "Proof of the Fundamental Theorem of Arithmetic"URL

    Read this proof of the FTA.

  - Massachusetts Institute of Technology: Dr. Srini Devadas and Dr. Eric Lehman's "Number Theory I"URL

    Read this lecture, which provides an introduction to decanting (see the *Die Hard* example on pages 5-7) and the Euclidean algorithm.

  - Harold Reiter's "Decanting"URL

    Read this paper. This is an easier version of this technique. Solutions to selected problems can be found here.

- **3.6.1: Another Look at the Division Algorithm**
  - The University of Western Australia: Greg Gamble's "Number Theory 1"URL

    Review the section on "Division Algorithm" again, and then attempt the 3 sample problems in the lecture.

- **3.6.2: Solving Ax + By = C over the Integers**

  - DavData: "Solving Ax + By = C"URL

    Read the brief text on solving $Ax+By=C$Ax+By=C.
  - Carnegie Mellon University: Victor Adamchick's "Integer Divisibility"URL

Read this lecture, which discusses solving an integer divisibility type of equation. You should focus on solving linear Diophantine equations. In particular, you should be able to find a single solution and then generate all solutions from the one you found.

Unit 4: Rational Numbers

In this unit, you will learn to prove some basic properties of rational numbers. For example, the set of rational numbers is dense in the set of real numbers. That means that strictly between any two real numbers, you can always find a rational number. The distinction between a fraction and a rational number will also be discussed. There is an easy way to tell whether a number given in decimal form is rational: if the digits of the representation regularly repeat in blocks, then the number is rational. If this is the case, you can find a pair of integers whose quotient is the given decimal. The unit discusses the mediant of a pair of rational fractions, and why the mediant does not depend on the values of its components, but instead on the way they are represented.

Completing this unit should take you approximately 9 hours.

- Upon successful completion of this unit, you will be able to:
  - find a pair of integers whose quotient is a given repeating decimal;
  - prove basic propositions about rational and irrational numbers;
  - recognize that the set of rational numbers is an ordered field, and use the properties of ordered fields to prove statements concerning rational numbers; and
  - prove that the set of rational numbers is not complete, but the set of real numbers is complete.

- **4.1: Fractions and Rational Numbers Are Not the Same**
  - Harold Reiter's "Fractions"URL

    Read this article. Pay special attention to the five problems on rational numbers at the beginning of the paper. Problem 10 will enable you to appreciate the different between the value of a number and the numeral used to express it. Pay special attention to Simpson's Paradox in the paper. Try the practice problems at the end of the reading. After you have attempted these problems, please check your answers here.

- **4.2: Representing Rational Numbers as Decimals**
  - MrCaryMath: "Rational vs. Irrational Numbers"Page

    Watch this video to learn about the difference between rational and irrational numbers.

  - Examplayer's "Recurring Decimals to Fractions"Page

Watch this video on the conversion of repeating decimals into fractions of the form abab with aa and bb integers.

- **4.3: The Existence of Irrational Numbers**
  - **4.3.1: The Square Roots of 2, 3, and 6 are All Irrational Numbers**
    - University of Missouri - Kansas City: "Proof: The Square Root of 2 Is Irrational"Page

      Watch this video, which shows a proof of the irrationality of the square roots of 2. Can you see how to use these ideas to prove that the square root of 3 and of 6 are also irrational?

    - The Math and Physics Channel: "Proof: Square Root of 3 Is Irrational"Page

      Watch this video, which shows a proof that the square root of 3 is irrational. After watching this video, do you think you could prove how the square root of 6 is also irrational?

  - **4.3.2: Density of Irrational Numbers**
    - New York University: Lawrence Tsang's "Real Numbers"URL
      Read pages 7 through 9, from "Density of Rational Numbers" through "Density of Irrational Numbers."

  - **4.3.3: Algebraic versus Transcendental Numbers**
    - Dan Sewell Ward's "Transcendental Numbers"URL
      Read this page, which describes transcendental numbers. Note that both ππ and ee are transcendental.

- **4.4: The Field of Rational Numbers**
  - New York University: Lawrence Tsang's "Real Numbers"URL
    Read pages 1-7 of the text. The first 6 pages discuss the field and order axioms for real numbers. The Completeness Axiom on page 6 is what distinguishes the rational numbers from the real numbers - the latter is COMPLETE, while the former is not.

Unit 5: Mathematical Induction

In this unit, you will prove propositions about an infinite set of positive integers. Mathematical induction is a technique used to formulate all such proofs. The term recursion refers to a method of defining sequences of numbers, functions, and other objects. The term mathematical induction refers to a method of proving properties of such recursively defined objects.

Completing this unit should take you approximately 4 hours.

- Upon successful completion of this unit, you will be able to:

- recognize the various forms and equivalences of mathematical induction;
- use mathematical induction to prove formulas for summations and products;
- use mathematical induction to prove inequalities; and
- use mathematical induction to prove properties about remainders.

- **5.1: Mathematical Induction Is Equivalent to the Well-Ordering Property of N**

  - The Mathsters: "Induction"Page

    Watch these videos, which provide informative discussions as to why the well-ordering principle of the natural numbers implies the principle of mathematical induction and discuss why the principle of strong mathematical induction implies the well-ordering principle of the natural numbers.

  - Julio de la Yncera's "Mathematical Induction"Page

    Watch this video, which provides an informative discussion on the principle of mathematical induction and the well-ordering principle of the natural numbers. It specifically addresses the notion of strong mathematical induction.

- **5.2: Proofs of Summations and Products**
  - **5.2.1: Sums and Products**

    - Khan Academy: "Proof by Induction"Page

      Watch this video, which examines how the principle of mathematical induction is used to prove statements involving sums and products of integers.

  - **5.2.2: Divisibility**

    - David Metzler's "Mathematical Induction, Divisibility Proof"Page

      Watch this video, which illustrates how the principle of strong mathematical induction can prove a statement about divisibility of natural numbers.

  - **5.2.3: Recursively Defined Functions**

    - Old Dominion University: Shunichi Toida's "Recursive Definition"URL
      - Recursive Definition
      - Recursive Definition of Function

      Read these essays. Notice the similarities between using recursion to define sets and using recursion to define functions. Then answer the four questions at the end of the first essay.

      In this type of definition, first a collection of elements to be included initially in the set is specified. These elements can be viewed as the seeds of the set being defined. Next, the rules to be used to generate elements of the set from elements already known to be in the set (initially the seeds) are given. These rules provide a method to construct the set, element by element, starting with the seeds. These rules can also be used to test elements for the membership in the set.

- MathDoctorBob: "Hard Inequality"Page

  Watch this video, which illustrates using the principle of strong mathematical induction to prove a statement about divisibility of natural numbers.

Unit 6: Relations and Functions

In this unit, you will learn about binary relations from a set AA to a set BB. Some of these relations are functions from AA to BB. Restricting our attention to relations from a set AA to the set AA, this unit discusses the properties of reflexivity(R), symmetry(S), anti-symmetry(A), and transitivity(T). Relations that satisfy R, S, and T are called equivalence relations, and those satisfying R, A, and T are called partial orderings.

Completing this unit should take you approximately 4 hours.

- Upon successful completion of this unit, you will be able to:
  - determine if a relation possesses certain properties, including reflexivity, symmetry, antisymmetry, and transitivity;
  - recognize and prove that certain relations are equivalence relations;
  - recognize and prove that certain relations are partial orderings and/or total orderings;
  - recognize when a relation is a function; and
  - determine when a function is one-to one and when it is onto.

- **6.1: Binary Relations on a Set A**
  - MathDoctorBob: "Binary Relations"Page

    Watch this video. It may be worth spending some time watching this video twice. The examples he provides exhibit several properties. These are the defining properties of an equivalence relation (see subunit 6.4) and Partial Ordering (see subunit 6.5).

- **6.2: Binary Relations from A to B**
  - **6.2.1: Relations that Are Functions**
    - Khan Academy: "Relations and Functions"Page

      Watch this video, which illustrates the notions of relations and functions. This video also provides examples of relations that are functions and some that are not.

  - **6.2.2: Injections, Surjections, and Bijections**
    - Khan Academy: "Injections, Surjections, and Bijections"Page

Watch these videos.

- **6.3: Equivalence Relations**
  - MathDoctorBob: "Binary Relations"Page

    Watch the last 10 minutes of this video again. It is especially important that you understand the relationship between an equivalence relation and the partition it induces.

  - Old Dominion University: Shunichi Toida's "Equivalence Relations"URL

    Read this page on equivalence relations. Then, answer the four questions at the bottom of the page.

- **6.4: Partial Orderings**
  - MathVids: "Equivalence Relations and Partial Orders"URL

    Watch this lecture. When you have finished, read the lecture notes and attempt the problems in the problem set.

Unit 7: Sets, Part II

In this unit, you will study cardinality. One startling realization is that not all infinite sets are the same size. In fact, there are many different size infinite sets. This can be made perfectly understandable to you at this stage of the course. In Unit 7, you learned about bijections from set AA to set BB. If two sets AA and BB have a bijection between them, they are said to be equinumerous. It turns out that the relation equinumerous is an equivalence relation on the collection of all subsets of the real numbers (in fact on any set of sets). The equivalence classes (the cells) of this relation are called cardinalities.

Completing this unit should take you approximately 8 hours.

- Upon successful completion of this unit, you will be able to:
  - compute the image and inverse image of sets under a given function;
  - determine if a function has an inverse function, and if so (and when possible), determine a formula for it;
  - establish bijections between certain subsets of real numbers, and connect this to computing the cardinality of these sets;
  - define the Cantor set, and prove that it is uncountable;
  - recognize countable subsets of the real numbers; and
  - establish properties of countable sets.

- **7.1: Cantor Diagonalization Theorem: The Existence of Uncountable Sets of Real Numbers**
  - Jeremy Lash, Matt Cerny, Michael Leahy, and Sumedha Pramod's "Cardinality"Page

Watch this video. Make sure you understand how two sets A and B can be equinumerous.

- American Public University: "Equivalent Sets", "Infinite Sets and Cardinality", and "Subset and Proper Subset"Page

  Watch these lectures to continue your studies on sets.

- **7.1.1: Proof of the Theorem**
  - Indian Institute of Technology, Madras: Arindama Singh's "Cantor's Little Theorem"URL

    Read this paper. Spend some time studying the proof of Cantor's Theorem on pages 3 and 4. Even though the proof is quite brief, this idea is new to you, and therefore is likely to be harder to understand.

  - University of Missouri, Kansas City: "Proof: There Are More Real Numbers than Natural Numbers"Page

    Watch this video to supplement the written proof of Cantor's Theorem.

  - Minute Physics: "How to Count Infinity"Page

    Watch this video about counting finite sets and Cantor's Diagonalization Theorem.

- **7.1.2: Even the Cantor Set Is Uncountable, the Base-3 Connection with the Cantor Ternary Set**
  - Elvis Zap's "The Cantor Set Is Uncountable"Page

    Watch this video to see a proof that the Cantor middle third set is uncountable.

- **7.1.3: Other Examples of Uncountable Subsets of R**
  - Brown University: Rich Schwartz's "Countable and Uncountable Sets"URL

    Read this document to learn about countable and uncountable sets. Focus on the several examples of uncountable subsets of R.

- **7.2: The Rational Numbers Are Countable**
  - University of Missouri, Kansas City: "Proof: There Are the Same Number of Rational Numbers as Natural Numbers"Page

    Watch this video to see a proof that rational numbers are countable.

- **7.2.1: The Proof**
  - Theorem of the Week: "Theorem 18: The Rational Numbers Are Countable"URL

Study the proof on this webpage, which shows that rational numbers are countable.

- ### 7.2.2: The Algebraic Numbers Are Countable
  - Alex Youcis' "Algebraic Numbers Are Countable"URL

    Study this proof, which demonstrates that algebraic numbers are countable.

  - University of St. Andrews: John O'Connor's "Infinity and Infinites"URL

    Read this page.

- ### 7.3: Other Bijections
  - Florida State University: Dr. Penelope Kirby's "Property of Functions"URL

    Please read the paper, paying special attention to examples 2.2.5, 2.2.6, and two functions a^xax and $\log_a x$logax in the paragraph above example 2.7.1.

Unit 8: Combinatorics

In this unit, you will learn to count. That is, you will learn to classify the objects of a set in such a way that one of several principles applies.

Completing this unit should take you approximately 11 hours.

- Upon successful completion of this unit, you will be able to:
  - distinguish between situations in which sampling with replacement is appropriate and those in which sampling without replacement should be used as well as the effect each has on the number of outcomes arising in various applications;
  - use the inclusion-exclusion principle to solve a variety of counting problems;
  - use the pigeon-hole principle in various settings; and
  - recognize when a counting problem requires the use combinations, permutations, or some other combinatorial object.

- ### 8.1: Counting Problems as Sampling Problems with Conditions on the Structure of the Sample
  - Indian Institute of Technology Madras: Kamala Krithivasan's "Permutations and Combinations"Page

    Watch this introduction to counting.

  - Harold Reiter's "Counting"URL

    Read this article. Attempt problems 1-20, starting on page 3. Once you have attempted these problems, check your answers here.

- **8.2: The Inclusion-Exclusion Principle**
  - CourseHack: "Inclusion/Exclusion"Page

    Watch these videos for for an introduction to the inclusion/exclusion principle and an example of the principle. Notice that the problem is about As, Bs, and Cs, not As, Bs, and Os as the teacher describes at the start.

  - **8.2.1: The Case with Just Two Sets**
    - Brian Veitch's "Formula for the Union of Sets - Two Sets and Three Sets"Page

      Watch this video, which provides an informative illustration of the addition formula for the cardinality of the union of two and three sets.

  - **8.2.2: The Proof**
    - Wikipedia: "Inclusion-Exclusion Principle"URL

      Read this article.

  - **8.2.3: Other Examples**
    - Course Shack: "Inclusion/Exclusion Examples"Page

      Watch these videos, which provide informative illustrations of the use of the inclusion-exclusion formula.

- **8.3: The Pigeon-Hole Principle (PHP)**
  - Dr. James Tanton's "Pigeon-Hole Principle"Page

    Watch this introduction to the pigeon-hole principle.

  - **8.3.1: The Standard Principle**
    - Indian Institute of Technology Madras: Kamala Krithivasan's "Pigeon Hole Principle"Page

      Watch the video, which provides a careful introduction to the pigeon-hole principle and provides several examples.

  - **8.3.2: Using the PHP Idea in Other Settings**
    - MathXpress: "Pigeon-hole Principle Problem Examples"Page

      Watch these videos. The first video provides some an application of the pigeon-hole principle to divisibility and modular arithmetic. The second video provides some applications of the pigeon-hole principle to operations involving integers.

    - Mr. T's Math Videos: "Basic Pigeon-Hole Principle Problems"Page

Watch this video, which provides some very elementary applications of the pigeon-hole principle.